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ABSTRACT

Reform efforts in mathematics aim to increase conceptual understanding, an aim that can be supported through concept maps. This study compared the conceptual knowledge of function held by college students in reform and traditional calculus sections at a large state university. Fourteen students from reform sections and 14 from traditional sections served as subjects. A primary task was the construction of a concept map of function. Four instructors of reform sessions and four from traditional sections also completed concept maps. Quantitative analyses of the concept maps showed that the core contents from both student groups matched poorly with instructors' core concepts. Qualitative analysis of the student maps revealed differences between the student groups, with the reform group using terminology common in the reform text and using fewer algorithmic references than the traditional group. The traditional group's maps contained more algorithmic references to hand-graphing techniques. Maps of both groups were considerably less well-structured than experts' maps and lacked their higher level categories. (Contains 9 figures, 10 tables, and 35 references.) (Author/SLD)

CONCEPT MAPS AS RESEARCH TOOLS IN MATHEMATICS

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ABSTRACT

Reform efforts in mathematics aim to increase conceptual understanding. Concept maps can serve well as an assessment instrument in this area. This study compared the conceptual knowledge of function held by college students enrolled in third quarter reform and traditional calculus sections at a large state research university. Fourteen students from reform sections and fourteen from traditional sections served as subjects. A primary task was the construction of a concept map of function. Eight PhD's in mathematics, four who taught reform classes and four who taught traditional classes, also completed concept maps of function. The study compared these expert maps with the student maps.

Quantitative analysis of the concept maps showed the two student groups' core concepts matched up poorly with the experts' core concepts. Neither group more closely matched the experts to a significant degree. Qualitative analysis of the core concepts and of the maps as a whole revealed differences between the student groups. The reform group used terminology common in the reform text and had fewer algorithmic references than the traditional group. The traditional students' maps contained more algorithmic references to hand-graphing techniques. The maps of both student groups were considerable less well structured than the experts' maps and lacked the higher-level categories found on the expert maps.

Introduction

In the past decade, much change has occurred in the field of mathematics education at both the K-12 and the undergraduate levels. Many consider the old pattern of teachers prescribing and students transcribing a generally ineffective teacning strategy for long-term learning, higher-order thinking, and versatile problem-solving. Publications such as *Everybody Counts* (National Research Council, 1989) and the *Curriculum and Evaluation Standards* (National Council of Teachers of Mathematics [NCTM], 1989) have led the way in K-12 curriculum reform. They have called for teaching methods that encourage students to construct their own mathematical knowledge and have emphasized understanding over the rote memorization of algorithms and the acquisition of mechanical problem-solving techniques.

This paper discusses one study of calculus reform in which concept maps played an important role as a research tool designed to capture information about conceptual understanding. First I present an overview of the research on conceptual understanding. Next, I delineate the development of the concept map and its variations. A description of the study follows along with the analysis of the subjects' concept maps. Finally, I discuss the value of the concept map as a research tool in mathematics and make suggestions for further applications and research.

Research on conceptual understanding

The NCTM Standards (1989) and other reform documents emphasize conceptual understanding. Cognitive psychologists seem to agree that the internal representation of knowledge resembles webs or networks of ideas that are organized and structured (Chi & Koeske, 1983; Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986; Janvier, 1987; Michener, 1978; Nickerson, 1985;

Novak, 1977: Pintrich, Marx & Boyle, 1993; Resnick & Ford, 1981: Royer, Cisero, & Carlo, 1993). The more connections between facts, ideas, and procedures, the better the understanding (Hiebert & Carpenter, 1992: Hiebert & Lefevre, 1986; Nickerson, 1985). Individuals whose knowledge is interconnected and structured activate large chunks of information when performing an activity in the knowledge domain (Fisher & Lipson, 1985; Prawat, 1989; Royer et al, 1993). A highly integrated knowledge structure hallmarks the transition from novice to expert performance (Royer et al, 1993).

In the current study I focused primarily on conceptual knowledge as opposed to procedural knowledge, and especially on understanding and meaningful learning as they apply to mathematics. Hiebert and Lefevre (1986) define "conceptual knowledge" as knowledge rich in relationships, a network in which the linking relationships are as prominent as the discrete pieces of information (p. 3). Conceptual knowledge grows in one of two ways: Joining two pieces of knowledge already in memory or adding a new piece of information to existing structures (Resnick and Ford, 1981). As described by Hiebert and Lefevre (1986), this construction of new knowledge occurs at two levels. A relationship connecting two pieces of information on the same level of abstractness constitutes the "primary" level relationship. The "reflective" level constructs relationships on a level of abstraction higher than that of the pieces of information they connect. This view promotes the idea that relationships in a knowledge network may be hierarchical.

Following the lead of Greeno (1978), this paper equates "learning with understanding" with "meaningful learning." A mathematical concept is "understood" when its mental representation becomes part of a network of representations (Hiebert & Carpenter, 1992). It follows that "a richness of knowledge is needed for deep understanding" (Michener, 1978, p. 374, see also

Nickerson, 1985). Conceptual knowledge, defined as a network of relationships can be acquired only in a meaningful way. At the opposite end of the continuum is rote learning, learning that exhibits few relationships and is closely tied to the context in which it is learned. Rote learning of inherently meaningful material often takes place when the student does not have sufficient background or time to construct a meaningful network of the information (Doyle, 1983; Novak, 1977).

Development of concept maps

While conceptual knowledge in mathematics is certainly desirable, how does one measure or assess such knowledge? This section briefly discusses the history and development of concept maps as tools for looking at conceptual knowledge.

Unlike assessment techniques used in heavily quantitative, primarily behavioral studies, cognitive techniques often employ both qualitative and quantitative measures. A cognitive approach seeks to determine the organization and structure of the knowledge base as well as the fluency and efficiency with which the knowledge can be used. Concept maps are an example of a direct technique used to look at the organization and structure of knowledge in this study. While Royer et al. (1993), in their review of techniques and procedures for assessing cognitive skills, do not list concept maps, other researchers acknowledge it as a "known technique" (Schoenfeld et al, 1993; see also Novak & Gowin, 1984; Shavelson, Ruiz-Primo, Lang & Lewin, in press).

In this study, I found concept maps preferable to other techniques for several reasons. First, concept maps maximize subject involvement and minimize the researcher's intrusive role. In drawing and labeling the linking lines, the subjects explicitly state the relationships they see. Other direct techniques require the researcher to infer what relationship the subject

perceived. In the same manner concept maps permit experts to organize their knowledge in their own way. This makes comparisons between student and experts more accurate and valid. While classifying concept into categories produces valid data for some domains, concept maps better depict mathematical knowledge and structure, which do not always lend themselves to simple categories and subcategories.

At present, the way one constructs and uses concept maps varies widely. In some instances, the subjects draw the maps; in other cases, the researchers construct the map from subject protocols. Some researchers require maps to be hierarchical; others do not. This diversity arises because concept mapping has emerged from two different theories. A short history of concept maps helps to explain these variations.

Novak and Gowin (1984), using the cognitive theory of Ausubel (1968), invented the schematic device they called a concept map. The concept map represents a set of concept meanings embedded in a framework of propositions. "Concepts" represent regularities in objects and events and are linked by words to form "propositions." Concept maps reflect the theory that conceptual knowledge forms a web or network of concepts and the connections between them. Novak and Gowin believe concept maps to be explicit, overt representations of the concepts and propositions a person holds, while acknowledging the difficulty of judging the degree of correspondence between the map and actual internal representation. Certainly subjects cannot draw concepts and links they do not possess in memory (unless they have memorized a concept map). Consequently, a concept map reveals, at least partially, their cognitive structure.

Because of their own work and their use of Ausubel's learning theory, Novak and Gowin (1984) posit that concept maps should be hierarchical, should

use linking words on all the conne ng lines, and should be constructed by the student. The emphasis on linking words and student construction is significant Since the complexity of the knowledge network directly relates to the degree of understanding, linking words reveal whether or not students correctly associate the concepts--or even if they link them at all. Having students personally construct the map provides more accurate data than having a researcher draw the maps using a student interview protocol (assuming the students are old enough to understand concept map construction).

Another non-hierarchical theory of cognitive structures developed about the same time as Ausubel's. Using word-association techniques based upon the theory of James Deese (1965), researchers represented concepts as nodes in a network but did not label the connecting lines. They used indirect methods (word associations, similarity judgments, and tree building) for eliciting representations. This school of thought evolved into semantic network theory where nodes are connected by directional and labeled lines to produce propositions. It proposed to account for the kinds of statements people were willing to make about a topic, as well as the relative speed with which they made them as measured in reaction time experiments (Resnick & Ford, 1981). However, the theory did not live up to its early promise (Johnson-Laird, 1983), since it failed to predict time differences within categories.

When one puts labels on the lines to produce propositions, semantic networks resemble Novak and Gowin's concept maps--with the exception that they do not have to be hierarchical (Shavelson et al., in press). The concept map theory, however, makes no claim about the rate of access to information. Studies by Fisher (Fisher, 1990; Fisher & Lipson, 1985) make it clear that some researchers have moved from an exclusively semantic network terminology to an

incorporation of Novak and Gowin's concept map terminology with references to Ausubel's theory.

Theoretical distinctions do exist for other researchers, particularly in regard to whether or not the knowledge representation must be hierarchical. Ausubel's (1968) theory of meaningful learning describes a hierarchical structure of conceptual knowledge, and those studies emanating from the Novak and Gowin tradition adhere strictly to the hierarchical requirement with the most general topic at the top of the map. White and Gunstone (1992) argue that whether or not a concept maps should be hierarchical depends on the structure of the subject matter. Researchers from the semantic network tradition tend toward a map with a general concept in the center and with links coming out much like the spokes of a wheel. Harnisch call them "spider maps" (Harnisch et al, 1994). In many cases the spider configuration can be redrawn to show a hierarchical relationship. However, because the question of hierarchy is not fully resolved. I did not assume or demand hierarchical structures in this study.

Seven recent studies used concept maps to assess cognitive structure or conceptual understanding (Beyerbach, 1986; Coleman, 1993; Laturno, 1994: Markham, Mintzes, & Jones, 1994; Park, 1993; Rogers, in press; Wallace & Mintzes 1990). They adhere to the Novak and Gowin tradition for constructing concept maps and thus require hierarchical maps. Two studies in particular (Markham et al. (1994); Wallace and Mintzes (1990)) address the use of concept maps as an assessment activity in science while two others (Laturno (1994) & Park (1993)) use concept maps in studies pertaining to mathematics.

Wallace and Mintzes (1990) looked at college students (n= 91) enrolled in an elementary science methods course. They examined the validity of concept mapping as an evaluation approach. Their conclusions support the concurrent

validity of concept maps as vehicles for documenting and exploring conceptual change in biology.

In a follow-up study, Markham et al. (1994) looked for more evidence of concurrent validity. Their subjects were advanced college biology majors (n = 25) and beginning nonmajors (n = 25). Again the authors concluded that their results showed further evidence for the concurrent validity of concept mapping as a research and evaluation tool in science education.

Conspicuous by their absence in the large number of studies using concept maps are studies in mathematics. Park (1993), in her evaluation of a new computer laboratory calculus course at the University of Illinois, does provide one example. She gave the students in both a traditional section and a computer section of the course a list of concepts to use in making hierarchical concept maps. Park scored the maps numerically and used a software program to compare the students' maps with a teacher's map. Park's concept map scores for the two calculus groups lists no significant differences in any scoring category or any total although the reform group's scores were generally higher and showed stronger congruence with the teacher's concept map. She also found a strong correlation between the concept maps scores and the post-achievement test scores (0.82). Park's qualitative analysis of the maps determined differences in the concepts each group added to its maps. The computer group gave many more visually-oriented terms, while the traditional group gave more application and technique-related terms. The linking words were equally low-level on all the maps.

Laturno's (1994) study sought to determine if concept maps corresponded with clinical interviews in determining concept connectivity and to see if concept maps could predict academic achievement. Her subjects (n=118) were community college students in self-paced, remedial mathematics courses.

Laturno followed the tradition of Novak and Gowin and instructed the students on hierarchical maps. She used a scoring scheme that gave points for additional concepts, relationships, levels of hierarchy, examples, and cross-links. Laturno's results indicate that student-generated concept maps show indications of validity as a research tool. The concept maps gave results comparable to both interviews concerning student knowledge of relationships between concepts and to the academic progression of students through the course.

Purpose of study

The work reported here is part of a larger study that included evaluation of concept maps as assessment tools. The study also compared students' from reform and traditional calculus classes conception of function. An example-nonexample questionnaire looked at students' concept image of function. This paper primarily documents the place of concept maps in the study. Williams (in press) reports other results and analyses.

A state university of over twenty thousand students served as the setting for this study. It is a top-tier research university and, as such, has high admission standards for its undergraduates. For the school year of this study (1993-1994), the university had two, three-quarter sequences of first-year calculus, the traditional and the reform. These sequences differed in at least four observable ways: 1) the textbook; 2) the technology required; 3) the types of problems assigned; and 4) the use of written group projects.

Two groups of subjects participated in the study. One group consisted of students enrolled in the reform and traditional calculus classes. The student subjects for the study were 28 volunteers enrolled in the third quarter of calculus at the large state university. Fourteen came from two reform sections; fourteen

came from a single traditional section. Each group (traditional and reform) had seven women and seven men. Each student completed two tasks: 1) drawing a concept map for the topic of function and 2) completing an example-nonexample questionnaire about function in an audio-taped interview.

Each student attended a session in which I presented instruction on concept maps. Some sessions involved a small group of students, while other times I met with a single student, depending on the students' availability. I explained concept maps in each session, showing them examples and stressing the importance of linking words on the lines. The examples included hierarchical maps, web or spider maps, and non-hierarchical maps. I told the students they could draw their maps however they wished. After the brief instruction, the students each drew practice maps using some concepts I gave them about fractions. When I was sure they could draw concept maps, I asked each student to brainstorm and come up with a list of terms related to function. After several minutes, I instructed them to draw a concept map for function using the terms they had generated as well as any others that might occur to them. No student interaction occurred in any group sessions. The subjects were able to work on the maps as long as they desired. All the students took less than an hour to complete the maps.

The other group of subjects included eight professors (PhD's in mathematics) at two different universities. Four taught at the large state research university the student subjects attended. The other four taught at a small, private, west-coast university. Two professors at each school taught the reform text and used graphing technology the year of the study. The other two professors at each school have taught calculus at this level, but only from traditional texts.

I gave the experts two concept map tasks. The "unrestricted" task was to draw a concept map for function from their perspective as a mathematician. The "restricted" task was to draw a concept map of function that represented what they would expect students completing the first-year calculus sequence to know. I met with each expert and explained concept maps to them in the same manner I did the students, showing them the same examples. I asked them to brainstorm to get their starting terms. I also gave the experts a 45-minute time limit to complete each map, as I felt this would help them focus on what could otherwise be considered an unlimited task. I did not stay with the experts while they worked on the tasks. Rather, I let them complete and return the maps at their convenience. They did not know what the second task was until after they had completed the first. Half of the experts did the restricted task first (one professor from the reform, one professor from the traditional at each school), while the other half completed the unrestricted task first.

Results and Analysis Using Concept Maps

Much of the analysis in this study focused on differences between groups of subjects, students as well as experts. Did concept maps reveal differences about the concept of function held by students in reform sections of calculus and in traditional sections of calculus? (For brevity and clarity, I will call these groups of students the "reform students" and the "traditional students.") I particularly looked for differences that might be attributable to different curricula that are clearly shown by concept maps. I also compared the concept maps of the two groups of experts. I looked to see if their knowledge about function, as exhibited in their concept maps, transcended any curricular differences or if they, too, may have been influenced by the curriculum they used.

Most researchers using concept maps have devised a scoring scheme to assign a numerical value to each map. The categories used for scoring often include valid propositions, levels of hierarchy, and cross-links. They occasionally include examples. After studying this set of data, I concluded it did not lend itself to a valid method of scoring. Several reasons support this conclusion.

First, the task I assigned--to create a concept map about function--was purposely unstructured for both the students and experts. I did not ask them to make a hierarchical map, nor did I tell them to organize related concepts together. I did not give them any concepts as examples, nor any hints as to what type of concepts I might expect. None of the students had any prior experience with concept maps, and only one expert said he had done concept maps. Consequently, while the maps reflect only the students' or experts' work and only their personal notions about functions, the maps proved to be widely divergent and complex and did not lend themselves to a numerical scoring scheme. For instance, the experts generated 197 different concepts on their unrestricted maps of function and 133 different concepts on their restricted maps. The students' maps produced just over 300 different terms. Figure 1 and Figure 2 show reduced copies of experts' maps. While the concepts and links are difficult to read, the overall complexity of the maps' structure is clearly visible. Figure 3, Figure 4, and Figure 5 are reduced copies of the maps for three students. These maps once again illustrate the diversity and complexity of some of the maps.

Some might argue that the diversity in the students' maps resulted because they did not understand the construction of concept maps. However, each student constructed a practice map, and I checked each one personally to be sure the student understood the basic procedure. All the student maps show

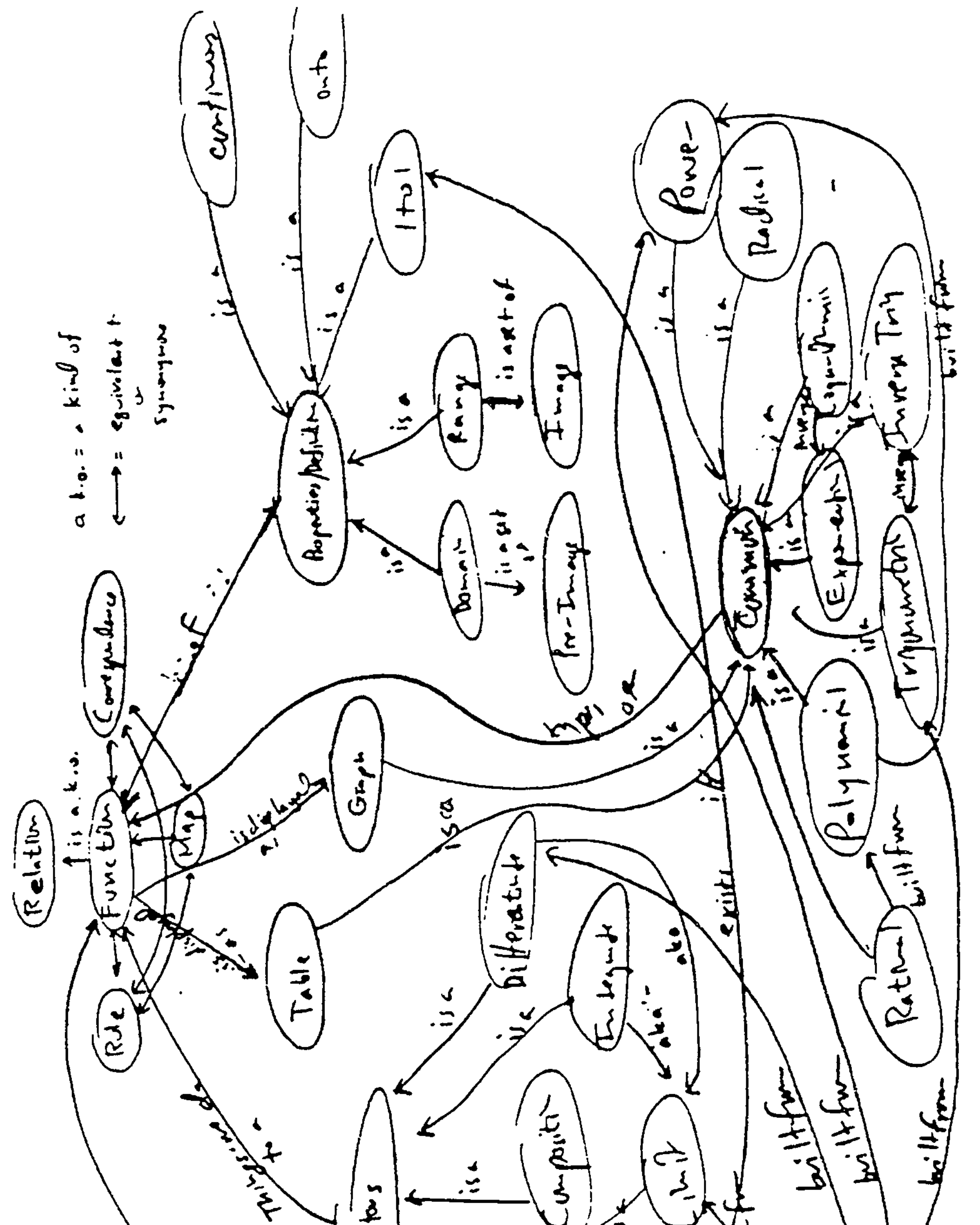

Figure 1: A Reform Expert's Concept Map

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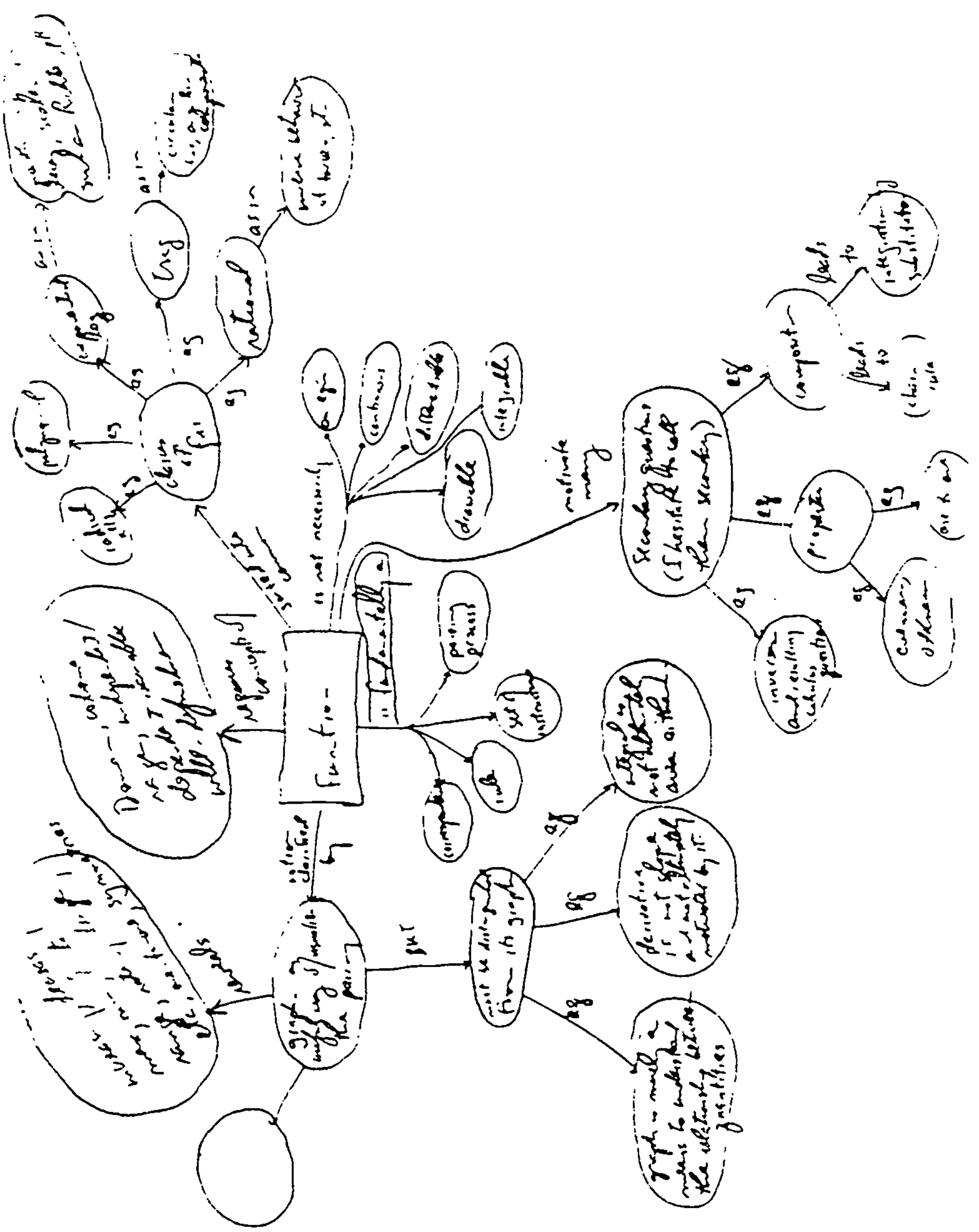

Figure 2 A Traditional Expert's Concept Map

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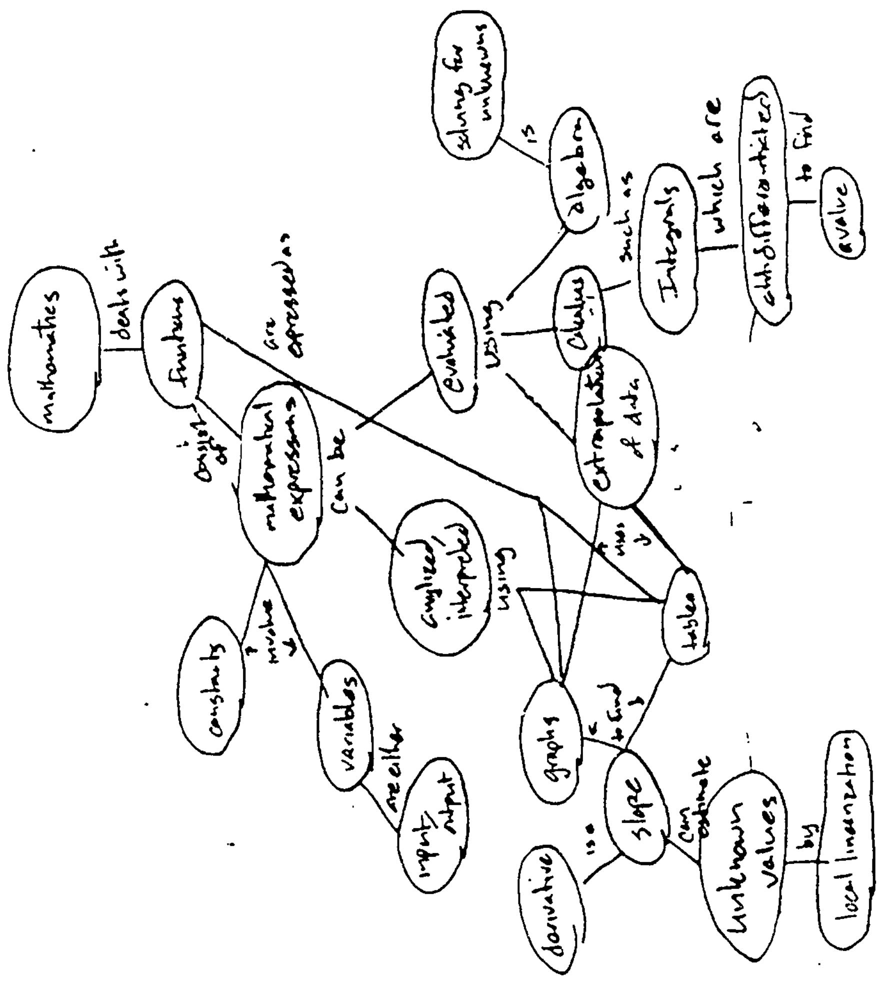

Figure 3: A Reform Student's Concept Map

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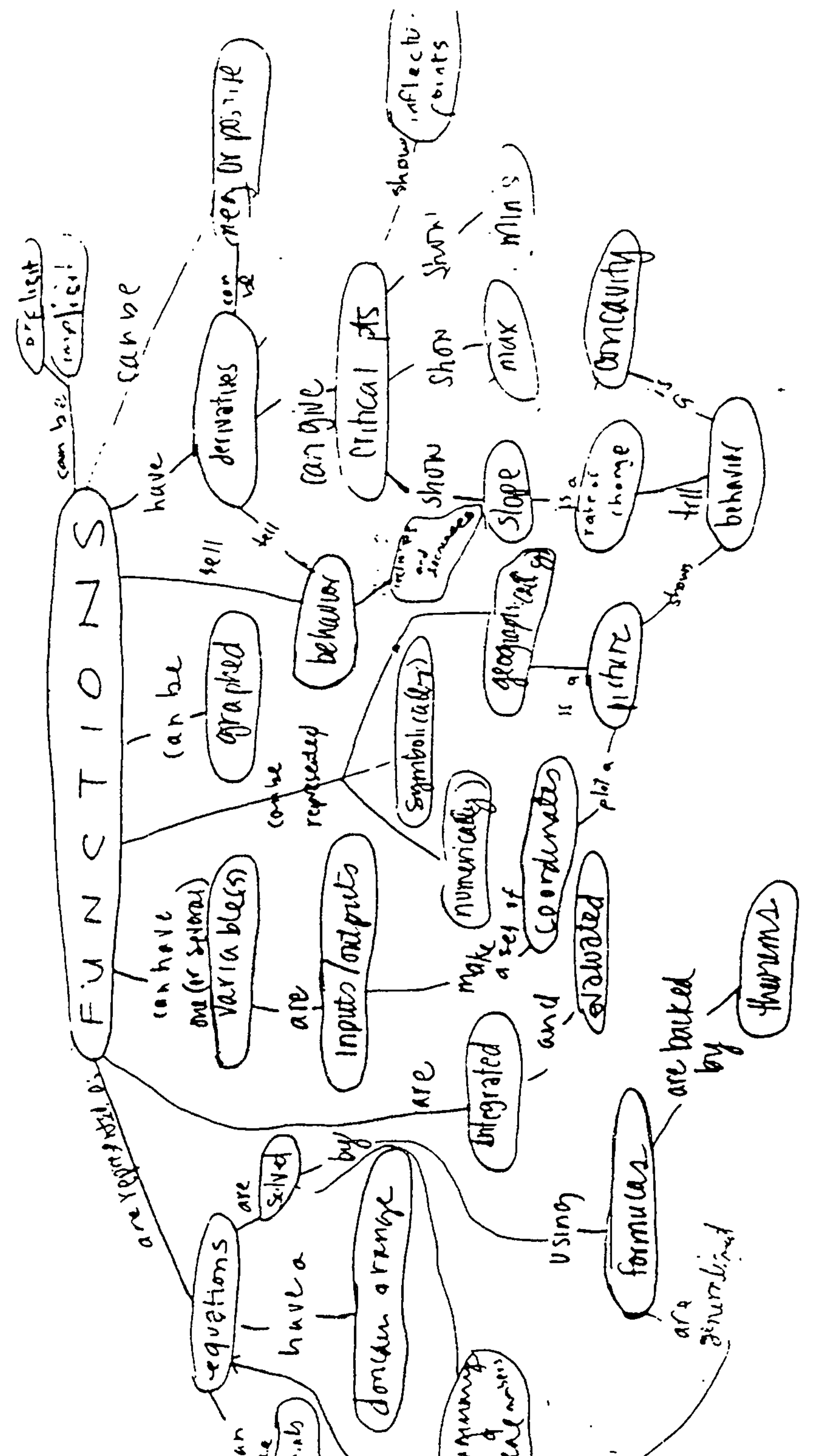

Figure 4: A Reform Student's Concept Map

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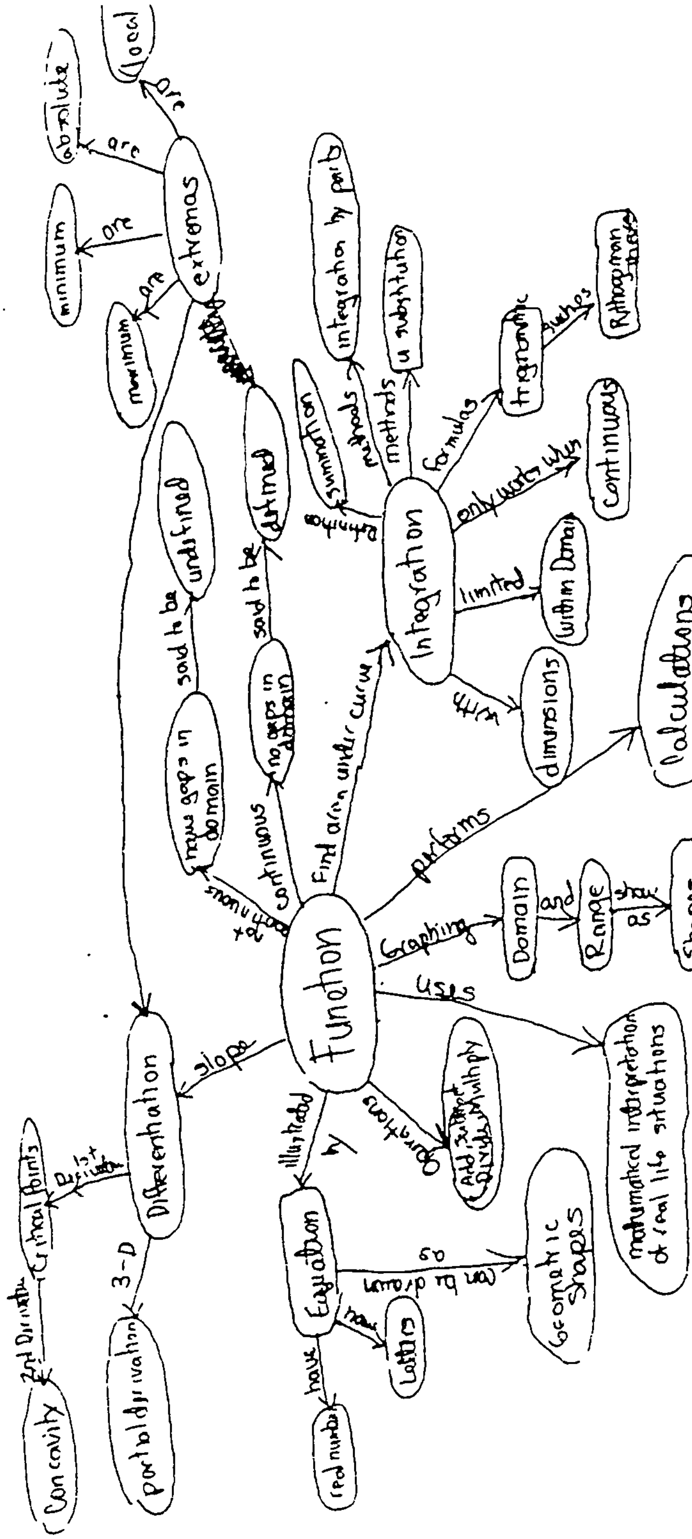

Figure 5: A Traditional Student's Concept Map

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that the students understood that concepts about functions went in the ovals while linking words denoting relationships went on the lines. Lacking any evidence of misunderstanding, one logically concludes that the wide diversity of the maps derives mainly from students' different conceptions about function, rather than from any difficulties they had with concept mapping.

Thus for this particular set of data, the numerical scoring schemes typically used do not appear to be valid. (For further discussion, see Williams, in press.) By developing lists of concepts most frequently used by the various groups of subjects, however, one can use the maps to make some valid quantitative comparisons.

Since the concept map task in this study was unstructured, many concepts emerged. In order to compare the students' concepts with those of the experts and to compare those of the student and expert subgroups with each other. I generated several "core" lists of concepts. Core lists consist of the concepts most frequently found on the concept maps of a given subject group. I detail the process below.

As mentioned, the experts each created two concept maps. For the restricted task, they constructed a map of what they felt students completing first-year calculus should know about function. For the unrestricted task, their map was to depict their view of function as a mathematician. I categorized the experts into three groups--reform experts. traditional experts, and combined experts. Since the three expert groups completed two concept maps, they generated six lists of concepts. I generated a core concept list under each of these conditions. The following sequence shows how I created three core lists for the restricted concept mapping tasl

To form a core list, I first compiled a list of the concepts used in the individual maps by each expert. Next I combined the reform experts' concepts to

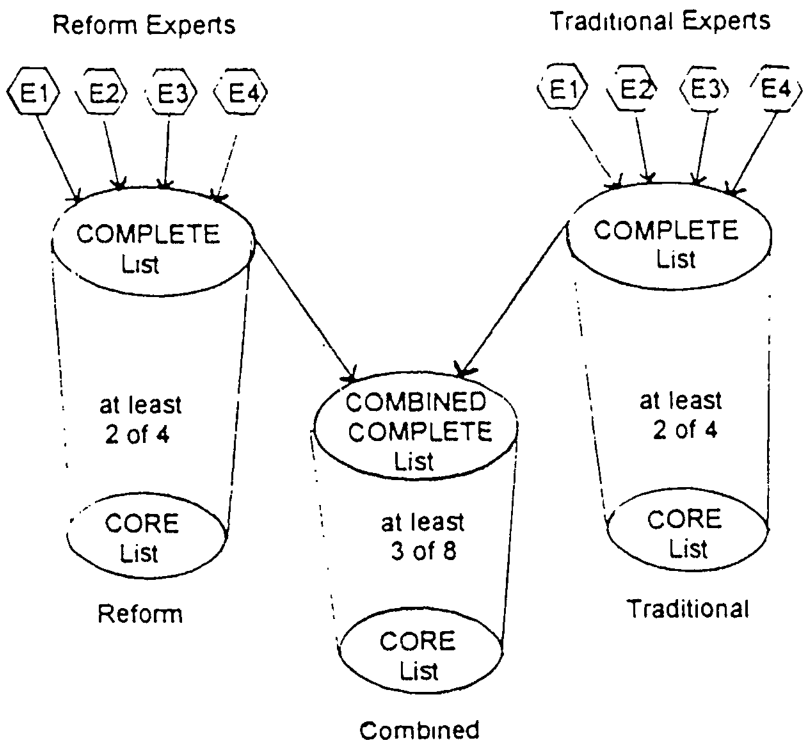

Figure 6: Formation of Core Lists

form a list--"reform experts' complete list" and the traditional experts' concepts to form "traditional experts' complete list" (see Figure 6). Putting these two lists together yielded the "combined experts' complete list"--a list of all terms used by the eight experts. If two or more reform experts (50%) used a concept on the reform experts' complete list, I put it on the "reform experts' core list." I used the same criterion for compiling the "traditional experts' core list." To form the "combined experts' core list," I chose all concepts that appeared on three of the eight experts' (38%) lists. (A natural break occurred at the 3-expert criterion with 25 out of 197 concepts used by a least three experts, 21 concepts used by exactly two, and 151 used by only one.) Using the same procedure and same

criteria. I generated the three core lists for the unrestricted task. The six core lists, along with the number of experts naming each concept, appear in Table 1.

The core lists for the restricted tasks were much longer than those for the unrestricted tasks, indicating the experts were in better agreement about the function concept as taught in first-year calculus than about function in general. The reform experts' core list for the restricted task contains 25 concepts, while for the unrestricted task the list has only 14. The traditional experts' core list for the restricted task has 23 concepts, while for the unrestricted task the list contains 8. Using the 3-out-of-8 criterion, the combined :perts' core list on the restricted task includes 25 terms, while its corresponding core list on the unrestricted has 10. These core lists served as benchmarks to which I compared the students' concepts.

I created core lists for the two groups of students in the same manner I had the expert core lists. I used a criterion of 7 out of 14 (50%) for the individual groups, reform and traditional, and a criterion of 10 out of 28 (36%) for the combined group. This process yielded three core lists: "reform students' core list", "traditional students' core list", and "combined student's core list" (see Table 2). They include 9, 8, and 10 concepts, respectively.

Analysis Using the Core Lists

Having compiled these lists, I was able to compare the various groups in several ways. The quantitative analysis primarily looked for agreement between the student groups' lists and the expert groups' lists and forms the first part of this analysis. The next part of the analysis details the qualitative differences between the two student groups and then between the students and experts The analysis section concludes with a qualitative comparison of the concept lists of the two expert groups.

Table 1

Experts' Core Concept Lists and Frequency Concepts Were Chosen: Restricted Task

Reform Experts' Core List	
composition	4
domain	4
exponential	4
function	4
graph/graphical	4
inverse	4
polynomial	4
range	4
trigonometric	4
derivative	3
integral/antiderivative	3
logarithmic	3
symbolic/equation	3
1-1	2
common/types	2
continuous	2
explicit	2
implicit	2
inverse trig	2
limit	2
onto	2
operations	2
rational	2
representations	2
table/numerically	2

Traditional Experts' Core List	
function	4
graph	4
continuous	3
differentiation	3
domain	3
range	3
1-1	2
composition	2
correspondence	2
derivative	2
symbolic/equation	2
even	2
exponential	2
increasing	2
integration/integrable	2
inverse	2
logarithmic	2
max/min/maximal	2
polynomial	2
properties	2
rate of change	2
trigonometric	2
variable	2

Combined Experts' Core List	
function	8
domain	7
graph/graphical	7
range	7
composition	6
exponential	6
inverse	6
polynomial	6
trigonometric	6
continuous	5
derivative	5
logarithmic	5
symbolic/equation	5
1-1	4
classes/common/types	4
differentiation/differentiate	4
integral/antiderivative	4
arithmetic operations	3
correspondence	3
integrate/integration	3
limit	3
properties	3
rate of change/change	3
rational	3
real world	3

Table 1 (continued)

Experts' Core Concept Lists and Frequency Concepts Were Chosen: Unrestricted Task

Reform Experts' Core List

function	4
chaotic	2
continuous	2
domain	2
function space	2
functor	2
graph	2
homomophism	2
isomorphism	2
linear	2
operator	2
range	2
recursive	2
table	2

Traditional Experts' Core List

function	4
1-1	3
domain	3
range	3
continuous	2
differentiable	2
graph	2
properties	2

Combined Experts' Core List

function	8
domain	5
range	5
1-1	4
graph	4
continuous	3
differentiable	3
function space	3
isomorphism	3
properties	3

30

31

Table 2

Student Groups' Core Concept Lists and Number of Students Choosing Each Concept

Reform Students' Core List		Traditional Students' Core List		Combined Students' Core List	
function	14	function	14	function	28
derivative	13	variables	11	derivative	22
graph	12	derivative	9	variables	21
slope	11	graph	9	graph	20
variables	10	integral	9	equation/ symbolically	15
equation/ symbolically	9	limit	7	integral/ integration	15
input	7	max/min	7	max/min	13
integral	7	slope	7	slope	13
line/linear	7			line/linear	10
				polynomial	10

I compared the students' core concept lists to the experts' core concept lists in two ways. First, I took each of the three student core lists and computed a ratio to reflect the number of students' concepts matching the experts' concepts to the total number of concepts on the *expert* list (see Table 3). For example, of the nine concepts on the reform students' core list, five of them are on the reform experts' core list (restricted task) that has 25 concepts, giving a ratio of 5 to 25 or 20%. (On Figure 7, this is the ratio of c to b.) Thus the

Table 3

Percentage of Experts' Core Lists Matched by Students' Core Lists

	Lists for Restricted Task			Lists for Unrestricted Task		
	Reform Experts	Traditional Experts	Combined Experts	Reform Experts	Traditional Experts	Combined Experts
No of Concepts on List	25	23	25	14	8	10
Reform Students	20%	26%	20%	21%	25%	20%
Traditional Students	20%	22%	20%	14%	38%	30%
Combined Students	24%	38%	20%	21%	38%	30%

Table 4

Percentage of Students' Core Lists That Matched Experts' Core Lists

		Lists for Restricted Task			Lists for Unrestricted Task		
	No of Concepts on List	Reform Experts	Traditional Experts	Combined Experts	Reform Experts	Traditional Experts	Combined Experts
Reform Students	9	56%	67%	56%	33%	22%	22%
Traditional Students	8	63%	63%	63%	25%	38%	38%
Combined Students	10	43%	57%	50%	29%	21%	21%

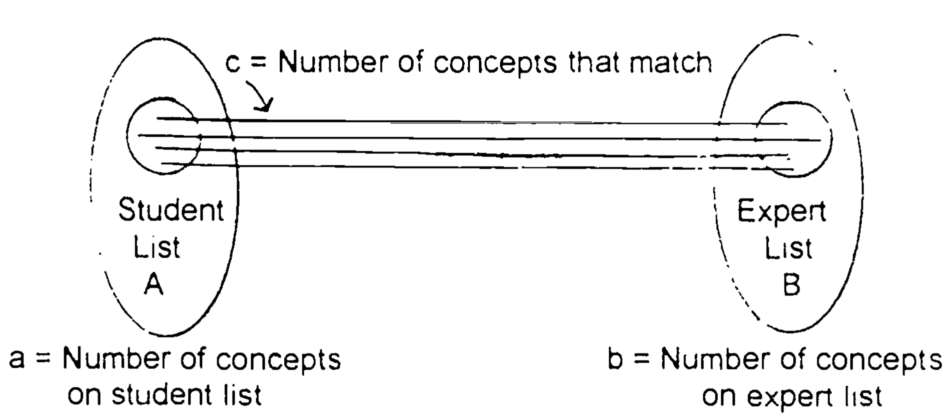

Figure 7: General Relationship of Lists and Number Concepts Used to Compute Comparison Percentages

percentages in Table 3 represent the portion of the experts' lists which the students also used.

This comparison yielded several observations. First, in general, the students' core lists do not match the experts' lists to a high degree. The correspondence is less than 40% for all comparisons, with most in the 20% range. Each of the three groups of students' core concepts matched the traditional experts' lists a little better than the other expert groups on both the restricted and unrestricted tasks. This may indicate the students know and use a higher percentage of concepts similar to those of the traditional experts. Although several percentages for the unrestricted task are higher than their corresponding percentages on the restricted task, one cannot conclude that the students' maps more closely resemble the experts' maps on the unrestricted task. One must take into account that the number of concepts on the core lists for the unrestricted task are much smaller. Consequently, a single concept can represent as much as 12%. Yet another comparison more accurately determines which expert mapping task that the students' maps most closely resemble.

I computed a second ratio: the number of students' concepts matching the experts' concepts to the total number of concepts on the *student* list (see Table 4). Again, using the same example, 5 of the 9 concepts on the reform students' core list match those on the reform experts' core list (restricted task), but this time the comparison is made to the nine-concept student list for a 5 to 9 or 56% ratio. (In Figure 7 this is the ratio c to a.) Thus Table 4 values show the percentage of student groups' core concepts that were also on the various experts' lists.

These figures show that a greater proportion of all the student lists match the experts' restricted task lists more closely than they match the experts' unrestricted task lists. The range for the restricted task comparisons is 43% to 67%, while for the unrestricted task, the range of match is only 21% to 38%. This result was expected, since the restricted task asked for what experts thought first-year calculus students should know.

Comparing core lists in this manner provided a quantitative measure by which to compare the concepts of the various groups. From a broad perspective, one can see that one-half to two-thirds of the students' core concepts are on the experts' core lists (restricted task) and yet they account for no more than 38% of any expert list, indicating disparity between the students and experts when it comes to concepts relating to function. Using this comparison, one also finds little difference between reform and traditional students' core concepts in their relation to the experts' core concepts.

Since creating composite lists such as the core lists may have masked some differences, I also looked at how the *individual* students' concepts compared to the experts' concepts on their core lists and their complete lists. I limited the comparison to the experts' restricted task, since it resembles the students' maps most closely.

In the first comparison, I determined for each student the number of concepts that matched each expert core and complete list. Using t-tests I found no significant differences between the two student groups in the number of concepts individuals used that matched concepts on the expert core concept lists.

In a manner similar to that described above, I found the percentage of each student's concepts that matched those on the experts' *core* lists (in Figure 7 this is the ratio c to b) as well as the number matching the experts' *complete* list of concepts (see Table 5). Looking at Table 5, one can see that the student lists corresponded better to the experts' core lists than to the experts' complete lists. Table 6 summarizes the comparisons using these individual match percentages. The mean percentage of match to the experts' core concepts ranged between 23% and 31%. The mean percentage of match to the experts' complete list of concepts ranged between 7.6% and 13.3%. Both the reform and traditional student groups matched the traditional experts' core and complete lists slightly better than they did the corresponding reform experts' lists. This finding is consistent with student core list comparisons reported above.

Another way in which I compared the student groups was to look at what percentage of each individual student's concepts was also on any of the experts' core or complete lists (in Figure 7 this is the ratio c to a). Table 7 shows the percentages for each student as well as the number of concepts each student map contained. In this comparison, the percentages for the match to the experts' complete lists are higher than those for the core lists. Since the combined experts' complete list consists of all terms used by the experts, Column H of Table 7 is a good indication of how meaningful and relevant the students' concepts are if one considers all expert concepts to be meaningful and relevant.

Table 5

Individual Student's Percentage Match to Expert Lists
Restricted Task

A	B	C	D	E	F	G
No of Concepts	25	23	25	97	75	133
R1	24%	26%	20%	11%	11%	10%
R2	24%	26%	24%	8%	11%	7%
R3	24%	26%	20%	9%	12%	8%
R4	24%	30%	24%	6%	13%	3%
R5	16%	26%	24%	9%	12%	8%
R6	32%	30%	32%	9%	11%	7%
R7	32%	39%	32%	12%	17%	13%
R8	20%	22%	16%	5%	8%	5%
R9	12%	17%	12%	6%	8%	6%
R10	48%	57%	40%	13%	21%	14%
R11	8%	13%	8%	3%	5%	3%
R12	20%	26%	20%	7%	11%	7%
R13	16%	22%	16%	7%	11%	7%
R14	28%	30%	28%	8%	13%	8%
T1	24%	30%	24%	7%	11%	6%
T2	28%	35%	28%	8%	17%	10%
T3	28%	35%	28%	9%	17%	11%
T4	28%	22%	28%	9%	11%	7%
T5	28%	43%	28%	7%	20%	11%
T6	8%	9%	8%	3%	7%	4%
T7	36%	39%	40%	12%	16%	11%
T8	40%	43%	36%	11%	20%	12%
T9	20%	26%	20%	5%	9%	5%
T10	16%	13%	16%	4%	7%	5%
T11	48%	57%	52%	14%	21%	13%
T12	20%	30%	24%	6%	13%	8%
T13	20%	26%	20%	6%	9%	6%
T14	16%	22%	16%	4%	8%	5%

Column Codes:

A: Student Designation
B: Match to Reform Experts' Core List
C: Match to Traditional Experts' Core List
D: Match to Combined Experts' Core List
E. Match to Reform Experts' Complete List
F: Match to Traditional Experts' Complete List
G Match to Combined Experts' Complete List

Table 6

Comparison of Percent Match of Individual Student's Concepts to Expert Lists--Restricted Task

Comparison to Reform Experts' Core Concepts

	Reform Students	Traditional Students
Mean	23.4%	25.7%
Variance	0.9%	1.1%
Range	8%-48%	8%-48%

Comparison to Traditional Experts' Core Concepts

	Reform Students	Traditional Students
Mean	28.0%	30.7%
Variance	1.0%	1.6%
Range	13%-57%	9%-57%

Comparison to Combined Experts' Core Concepts

	Reform Students	Traditional Students
Mean	22.6%	26.3%
Variance	0.7%	1.2%
Range	8%-40%	8%-52%

Comparison to Reform Experts' Complete List

	Reform Students	Traditional Students
Mean	8.3%	7.7%
Variance	0.1%	0.1%
Range	3%-13%	3%-14%

Comparison to Traditional Experts' Complete List

	Reform Students	Traditional Students
Mean	11.7%	13.3%
Variance	0.2%	0.3%
Range	55-21%	7%-21%

Comparison to Combined Experts' Complete List

	Reform Students	Traditional Students
Mean	7.6%	7.9%
Variance	0.1%	0.1%
Range	3%-14%	4%-13%

Table 7

Percentage of Individual Student's Concepts That Matched Expert Lists (Restricted Task)

A	B	C	D	E	F	G	H
R1	22	27%	27%	23%	50%	36%	59%
R2	31	19%	19%	19%	26%	26%	29%
R3	21	29%	29%	24%	43%	43%	52%
R4	29	21%	24%	21%	21%	34%	14%
R5	26	15%	23%	23%	35%	35%	42%
R6	21	38%	33%	38%	43%	38%	43%
R7	27	30%	33%	30%	44%	48%	63%
R8	17	29%	29%	24%	29%	35%	41%
R9	23	13%	17%	13%	26%	26%	35%
R10	32	38%	41%	31%	41%	50%	59%
R11	29	7%	10%	7%	10%	14%	14%
R12	22	23%	27%	23%	32%	36%	41%
R13	22	18%	23%	18%	32%	36%	41%
R14	36	19%	19%	19%	22%	28%	31%
T1	31	19%	23%	19%	23%	26%	26%
T2	44	16%	18%	16%	18%	30%	30%
T3	41	17%	20%	17%	22%	32%	34%
T4	20	35%	25%	35%	45%	40%	45%
T5	28	25%	36%	25%	25%	54%	54%
T6	19	11%	11%	11%	16%	26%	26%
T7	35	26%	26%	29%	34%	34%	40%
T8	29	34%	34%	31%	38%	52%	55%
T9	17	29%	35%	29%	29%	41%	41%
T10	12	33%	25%	33%	33%	42%	50%
T11	36	33%	36%	36%	39%	44%	47%
T12	45	11%	16%	13%	13%	22%	22%
T13	15	33%	40%	33%	40%	47%	53%
T14	24	17%	21%	17%	17%	25%	25%

Column Codes:

A: Student Designation
B: Number of Concepts Generated by Student
C: % Student's Concepts on Reform Experts' Core List
D: % Student's Concepts on Traditional Experts' Core List
E: % Student's Concepts on Combined Experts' Core List
F: % Student's Concepts on Reform Experts' Complete List
G: % Student's Concepts on Traditional Experts' Complete List
H: % Student's Concepts on Combined Experts' Complete List

Comparison of the two student group's number of concepts generated using a t-test revealed no significant difference. Although the traditional students generated more concepts on average (28.3 concepts was the mean for

the traditional students, 25.6 the mean for the reform). this difference was not significant at the 0.05 level. Table 8 summarizes the data about the percentage of the students' concepts that matched the experts'. In looking at the proportion of the students' concepts that were on the experts' complete lists. each group matched its corresponding expert group to a higher degree than its peers.

This analysis generally agrees with the insights gleaned from the student core list comparisons. It is useful, however, because comparing the students' individual percentages of concepts that are on the expert list approximates assigning each map a numerical score for number of relevant and meaningful concepts.

The data presented in Tables 7 and 8 also provides further evidence of how difficult it would be to score the students' concept maps numerically. If one assumed that a concept mentioned by any expert was meaningful, the majority of the students' concepts (which did not match those of experts) would have to be judged meaningless or irrelevant. Can a *proposition* be considered meaningful if it contains one or more concepts that are not meaningful? How would one assign such "partial credit?" How would one judge levels of hierarchy if some of the propositions in the chain were invalid? Once again, scoring this data presents serious problems that might be solved arbitrarily but never satisfactorily.

In summary, one can draw several important conclusions from quantitative analyses of the concept lists: 1) In general, the core concepts of the reform students' lists matched the experts' lists to the same degree that traditional students' lists did. Student lists from both groups matched traditional experts slightly better than reform experts. 2) Both student groups' core concepts matched a relatively low percentage of the experts' lists. 3) When considering the proportion of the *individual student's* concepts that matched

Table 8

Comparison of Portion of Individual Student's Concepts That Matched Expert Lists--Restricted Task

Comparison of Number of Concepts Generated

	Reform Students	Traditional Students	t-Statistic
Mean	25.6	28.3	-0.838
Variance	28.1	118.7	
Range	17-36	12-45	

Comparison to Reform Experts' Core Concepts

	Reform Students	Traditional Students	t-Statistic
Mean	23.3%	24.3%	-0.295
Variance	0.8%	0.8%	
Range	7%-38%	11%-35%	

Comparison to Traditional Experts' Core Concepts

	Reform Students	Traditional Students	t-Statistic
Mean	25.4%	26.0%	-0.186
Variance	0.6%	0.8%	
Range	10%-33%	11%-40%	

Comparison to Combined Experts' Core Concepts

	Reform Students	Traditional Students	t-Statistic
Mean	22.3%	24.6%	-0.738
Variance	0.5%	0.8%	
Range	7%-38%	11%-36%	

Comparison to Reform Experts' Complete List

	Reform Students	Traditional Students	t-Statistic
Mean	33.2%	28.0%	1.253
Variance	1.2%	1.1%	
Range	10%-44%	13%-45%	

Comparison to Traditional Experts' Complete List

	Reform Students	Traditional Students	t-Statistic
Mean	33.7%	36.7%	-0.539
Variance	0.9%	1.1%	
Range	14%-50%	22%-54%	

Comparison to Combined Experts' Complete List

	Reform Students	Traditional Students	t-Statistic
Mean	40.3%	39.2%	0.212
Variance	2.3%	1.4%	
Range	14%-79%	22%-55%	

those on experts' list. the reform students and traditional students performed about the same. 4) The individual students' proportion of match to the experts' complete list of concepts is an indicator of the number of meaningful and relevant concepts the students used. As such it provides a way to numerically compare a portion of the students' concept maps.

Qualitative Analysis of Core Concepts

While quantitative comparisons provide information about how many or what proportion of concepts are the same or different for various groups, qualitative analysis looks for substantive similarities and differences. I qualitatively compared the experts' core concept lists with the students' core lists with interesting results. A similar analysis of the concepts with respect to the students' core lists did highlight differences in the two student groups. The final comparison in this section looks for differences between the reform and traditional experts.

The quantitative analysis of the concept lists revealed low correspondence of students' concepts with experts' concepts. Comparing students' concepts with experts' concepts on a qualitative basis sheds some light on why this disparity exists. Since earlier analysis showed little difference between the reform and traditional students' agreement with the experts, I used the combined experts' core concepts and the combined students' core concepts for the qualitative comparison. As has been stated, the experts produced 25 core concepts, the students only 10 (see Table 9). Six of the 10 student concepts are on the experts' list. Of particular interest, however, are certain items on the experts' list that seldom appear on any student's map.

One group of concepts from the experts' list--**domain**, **range**, and **correspondence**--relates to the definition of a function. None of these terms

Table 9

Combined Core Concept List for Experts and Students

Combined Experts' Core Concepts
function
domain
graph/graphical
range
composition
exponential
inverse
polynomial
trigonometric
continuous
derivative
logarithmic
symbolic/equation
1-1
classes/common/types/familiar
differentiation/differentiate
integral/antiderivative
adding/dividing/arithmetic operations
correspondence
integrate/integration
limit
properties
rate of change/change
rational
real world

Combined Students' Core Concepts
function
derivative
variables
equation/symbolically
integral/integration
max/min
slope
line/linear
polynomial

Concepts given in descending order by frequency of use

appears on the students' combined core list (nor on the other two student core lists). Seven of the eight experts (88%) listed **domain** and **range**, indicating that these concepts form an integral part of their view of function. This does not hold true for the students.

Another group of concepts on the experts' combined list fits together nicely. **Exponential**, **polynomial**, **trigonometric**, **logarithmic**, and **rational** are all **classes** or **common types** of functions. Only **polynomial**, the simplest and most pervasive function in high school algebra, made the students' combined list, although undoubtedly the students have encountered all these different types of functions. **Trigonometric** and **rational** appeared only on student maps from the traditional classes (and rational was sometimes used to mean rational number, not rational function). Only two students (7%) seemed to set up a class or type grouping of functions on their concept maps. One student had **polynomial** and **rational** as her two branches, another had **linear**, **exponential**, and **quadratic** as his. Five of the experts (63%) had such a grouping on their maps.

A third group emerging from the experts' core list involves **properties** of functions, **1-1, continuous, differentiable,** and having an **inverse.** Thirty-eight percent of the experts had such a group. Once again, none of these concepts figured on the students' combined list. Only one student from the traditional group used **1-1**, **continuous**, and **inverse**, while four others listed **continuous** alone. One student listed **inverse.** No student used **differentiable**.

The experts' list yielded a fourth cluster of concepts: operations one performs on functions. These include **composition, differentiation, integration,** and combining with **arithmetic operations.** Determining the **inverse** of a function might also be considered a part of this group. Only one student used **differentiation** or **differentiate**; four used **integrate** or **integration**; two mentioned **composite**; and one said functions could be **added, subtracted, multiplied, or divided**. No student concept map showed any indication of an operations group, while four of the eight (50%) expert maps showed a strong operations grouping. To summarize this comparison of the

Table10

Concepts Used Primarily by One Student Group

Concepts Used by Reform Students
behavior (3)*
contour lines (2)
evaluated (3)
global (2)
implicit (2)
increasing/decreasing (3)
level sets (2)
local linearity (3)
numerically (3)
real life (3)
slope of tangent (3)
symbolically (4)
table (5)
understanding (2)
input/output (7 of 9)**
geometrically (2)
algebraically (1)

Concepts Used by Traditional Students
absolute (4)
asymptotes (4)
defined (6)
rational (5)
real numbers (3)
trigonometric (3)
undefined (4)
volume (4)
domain (6 of 8)
numbers (6 of 8)
shape (3 of 4)
f(x) (3 of 4)
velocity (4 of 5)
range (5 of 7)

*3 students from this group are the only students who used this concept.

**7 of the 9 students using the concept were from this group.

experts and students, one can say that the experts' core lists showed higher-level groupings that are not present in the students' concepts.

The three student core lists are very similar (see Table 2). The differences between the two student groups pertain to concepts that did *not* make the core lists. Table 10 shows the concepts used only by one group of students or primarily by one group. The resulting sets of concepts illustrate the

distinctions in the two teaching methods. For example, **asymptotes**, **defined**, **rational**, **real numbers**, **undefined**, and **domain**--concepts used only by traditional students--all relate to graphing by hand. To graph a **rational** function, one must determine which **real numbers** must be excluded from the **domain**. At the values for which the denominator is zero, the function is **undefined,** and an **asymptote** may exist. **Velocity** and **volume**, again terms used by traditional students, represent typical applications used in traditional texts for derivatives and integrals. On the other hand, the concepts from the students in the reform group strongly reflect the emphasis and terminology of the reform text. It presents topics "**geometrically**, **numerically**, and **algebraically**" (Hughes-Hallett et al., 1992, p. v). Only the reform students used these terms. Reform students call the numerical form **tables**. Instead of being concerned with graphing by hand, reform students stress with the **behavior** of the functions--where they are **increasing** or **decreasing**. One can discuss the **slope of a tangent** in terms of **local linearity**. The examples reform students use come from **real life** and promote **understanding**. **Input/output** terminology in the reform classes replaces the **domain/range** language of the traditional classes. Even though the reform subjects had experienced a traditional approach to functions in their high school algebra classes, their distinctive terminology indicates that they are assimilating, at the very least, the vocabulary used in the reform calculus classes and textbook. It may also point to a difference in the content being covered in the two classes.

A comparison of the experts' core concepts completes this section. For the restricted task, the combined experts' core list contains 25 terms (see Table 1). Fifteen of the terms (60%) are on both the reform experts' core list and the traditional experts' core list. I looked at the concepts that were only on the reform experts' core list to determine any groupings that might distinguish them

from the concepts only on the traditional experts' core list. No strong differences emerged. The reform experts' core list had two terms, **representations** and **table**, that are emphasized more in reform classes. The traditional experts' core list contained **max/min** and **increasing**, concepts used primarily by the traditional students. Terms from the professors' different curricula do distinguish the core lists, but only minimally considering the size of the lists.

On the experts' unrestricted concept mapping task, the eight professors generated 197 different concepts. The combined experts' core list contained only 10 concepts, showing the great diversity of these maps (see Table 1). All eight of the traditional experts' core concepts were on the combined experts' core list. The reform experts' core list had 14 concepts, only 7 of which are on the combined experts' core list. The other seven concepts relate more to higher mathematics than to the reform curriculum.

In summary, qualitative analysis of the lists of concepts gleaned from the concept maps did show distinctions between the two student groups, differences in terminology that linked each group to its respective curriculum. The qualitative analysis also reflected differences between the experts and the combined student group. The experts' combined core of concepts shows at least four higher-level categories--definition, class or type, properties, and operations--all of which are virtually non-existent in the students' maps. A qualitative study of the experts' core lists did not reveal any major differences along curricular lines.

General Analysis of the Concept Maps

Since this set of data did not lend itself to further quantitative analysis, the qualitative study of the concept maps took on added importance. Once again, I looked for differences between the two students groups that would correlate to the differences in their curricula or to their respective experts. A study of the

maps general appearance and structure did yield insights about the two student groups and about the experts. This sections details those findings.

The most striking observation about the students' maps, and this holds for both reform and traditional groups, is that many of their concepts and propositions were trivial or irrelevant. For example, many of the maps showed an emphasis on **variables**, even listing **x**, **y**, and **z** as concepts. Another student termed them "**letters**" and has three concepts under letters: **a-z**, **Greek**. and **x's** and **y's**. Thirteen of the 14 students from the traditional group have some reference to **variables** on their maps, as do 12 of the 14 students from the reform classes. Two students from each group also connected **slope** to **rise over run**, a concept taught in first-year algebra and far from the topic of function. Often the students list concepts closely tied to the types of exercises they did in class, such as finding **maxima** and **minima** and classifying them as "**absolute**" ("**global**" for the reform classes) or "**local**." One could say their preoccupation is with the "trees" and not the "forest."

Some maps had sections that completely veered from the topic. One might hypothesize the students were simply trying to fill up the page or use up the time. Figure 8 shows half of a map on which there are no actual mathematical terms (the other half was little better). Another map contains the propositional chain "**solutions** help to get **good grade** may produce **job** lead **$** pursue **happiness**." Figure 9 shows one student's complete map. Were it not for the concepts **integrals** and **derivatives**, one would not know this student had been in a calculus course.

The second most noticeable characteristic of the students' maps was their algorithmic nature, particularly among the students from traditional classes. By algorithmic nature, I mean that instead of giving *concepts* and the *relationships* among them, the students gave *steps* in a *procedure*. For example, one student

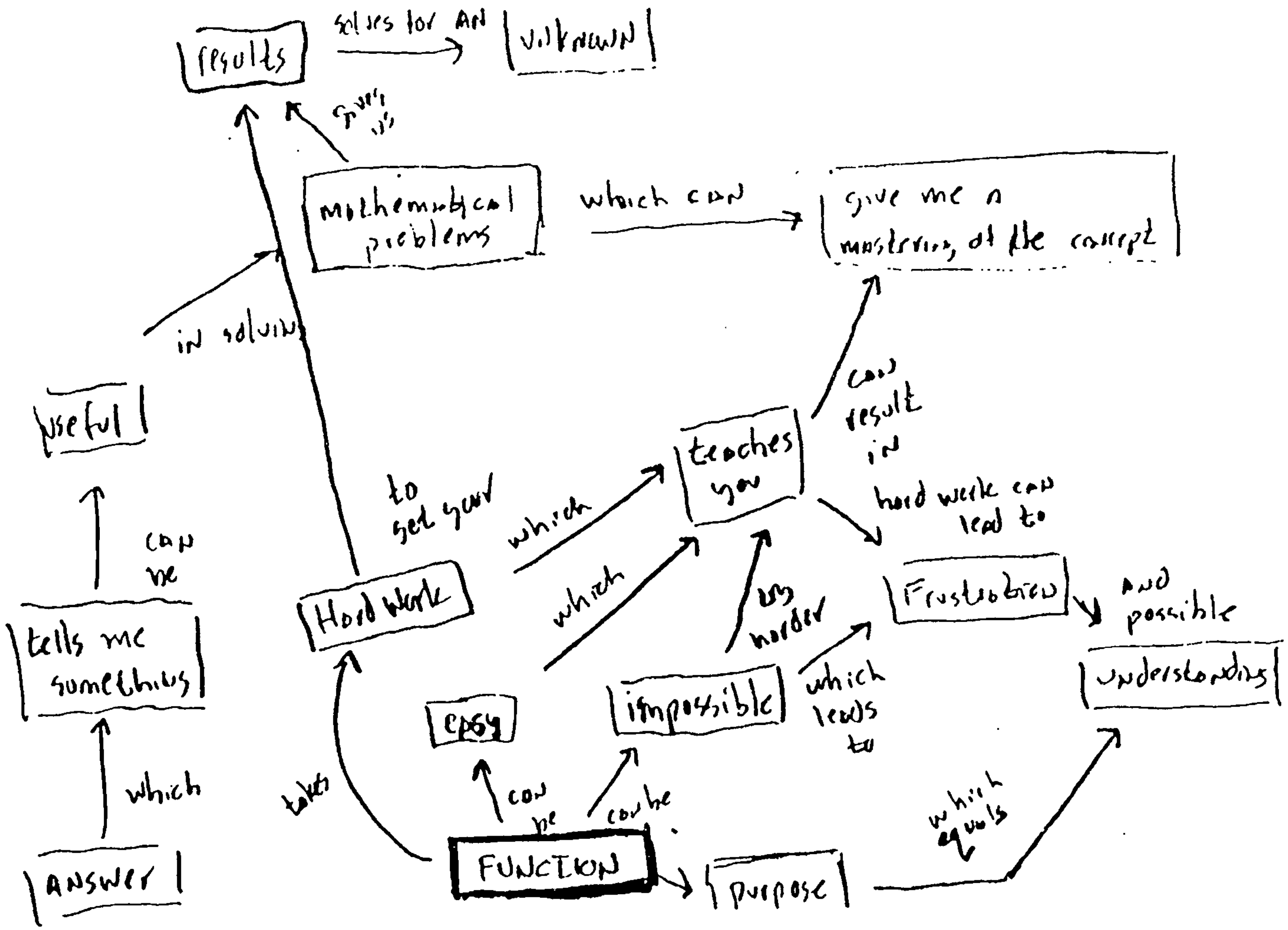

Figure 8: Portion of a Student's Concept Map with Few Mathematical Terms

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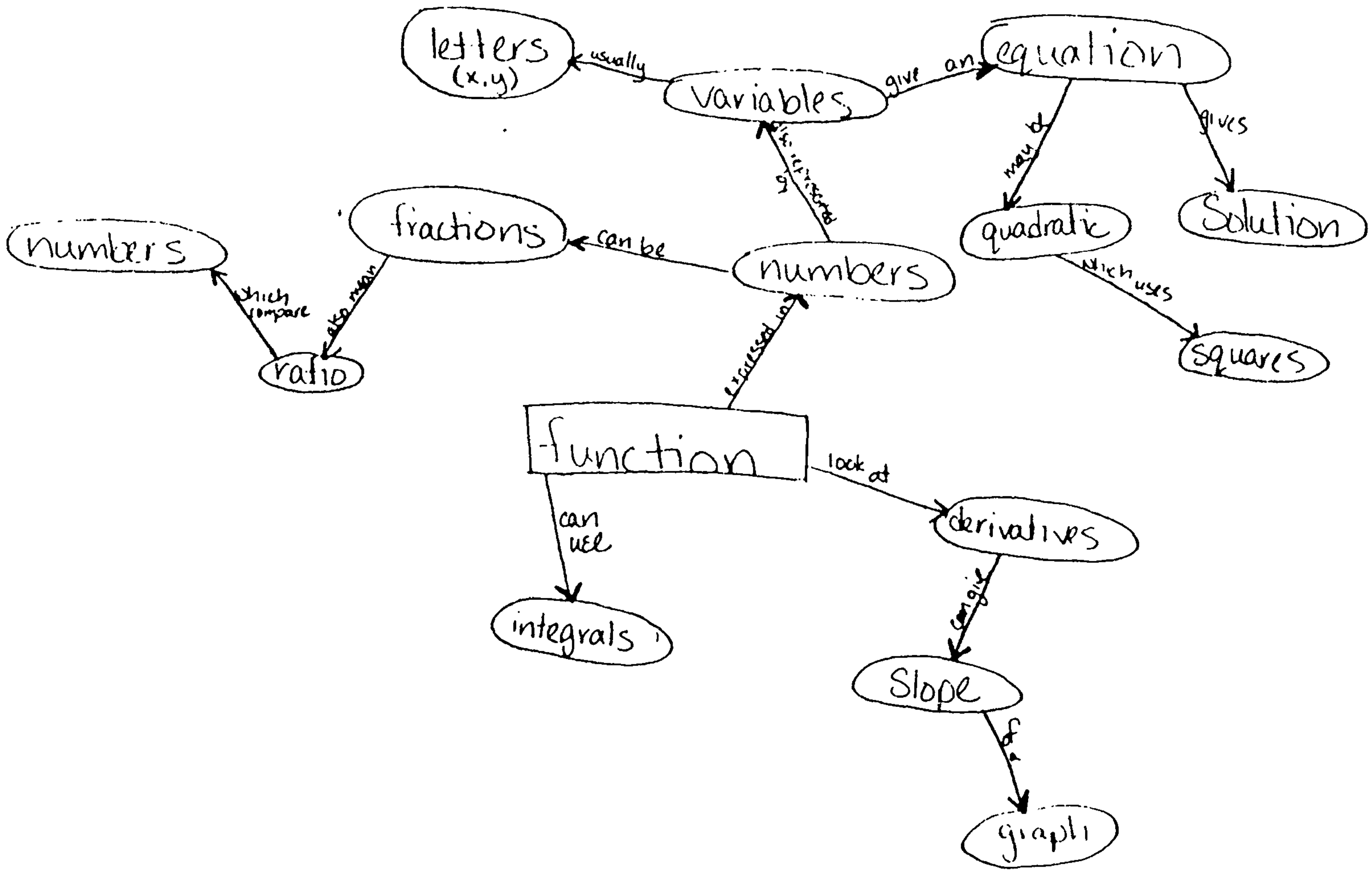

Figure 9: A Student's Complete Concept Map Showing Few Calculus Terms

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has the chain: "**function** can be **continuous** if **you can draw without lifting your pencil** or **they have no undefined points** such as **f(x)=1/x at x = 0** is **a hole at x = 0** which is **incontinuous**." She starts out with the concepts of **function** and **continuous** but then drifts into procedural steps. The algorithmic portions of the maps can usually be identified by the nature of the linking words, such as "by plotting" or "by locating," or by long phrases in the concept ovals, such as "it passes the vertical line test if $f^{-1}(y) = x^{-1}$."

Seven of the traditional students (50%) had groupings where algorithms or processes are evident, while only four (29%) from the reform classes did. The two most extensive maps from the traditional group show heavy algorithmic influence and could almost serve as procedural outlines of chapters in a traditional text. No map from the reform group showed extensive algorithmic groupings.

The reform text regularly presented functions in different representations I studied the student maps to determine, if possible, a student's predominant view of function--whether it was an equation, a graph, a set of ordered pairs, or perhaps something else. The linking words played an important part in this analysis. For instance, one student's map has these propositions: "**function** <u>consists of</u> **variables**," "**function** <u>can be</u> **graphed**," "**function** <u>can be</u> **polynomial**." Using the same link for **graphed** and **polynomial** indicates the same connection: possibility, not necessity. These propositions, along with heavily algorithmic portions of her map, indicate an equation view of function. Other maps I judged to have an equation perspective had **graph** separated from **function** by other propositions. Some students said straightforwardly, "**function** (*sic*) are **equations**."

Three students from the traditional group (21%) and four from the reform group (29%) made a connection between function and real life situations, but

from the two groups one gets a different sense of the relationship. The three students from the traditional group had these propositional chains: "**functions** serve to **represent complex problems, e.g. velocity**," "**function** uses **mathematical interpretation of real life situations**," "**equations** discover **natural phenomena** links **velocity and acceleration** example **balls falling in air**." Velocity and acceleration are the typical examples in a traditional text. The least extensive example from the reform group contained the chain "**function** is found in **the real world** like **economics and engineering and medical field**." Another reform student connects three chains and conveys the important concept of using functions to make predictions from collected data. A third student from the reform group shows that functions serve as approximations to real-life situations. A fourth map from the reform group has these propositional chains: "**functions** don't always involve **equations** some are about **real life situations** an example **death rate of a population as a function of time**," and "**functions** involve **modeling** an example **exponential decay**--an example **interest rate for a savings account**." While the evidence is not conclusive, the concept maps indicated that students from the reform group had a better understanding that functions may be used to model actual, real-life situations.

I also examined the maps to see if they reflected knowledge about the definition of function. In the group from the reform classes, one student has **domain** and **range** as concepts. Four others have **input** and **output**. None of the reform students indicated that each element of the domain can be paired with only a single range element, an essential part of the function definition. In the group from the traditional classes, four simply listed **domain** and **range** as concepts. Three others used **domain** and **range** and included the requirement about unique values for the range elements on their maps.

Did the maps reveal any differences in hierarchy and integration of concepts between the two groups? In a word, no. Few maps showed any significant hierarchical structuring, although I did show them examples of hierarchical maps. The number of concepts emanating directly from **function** ranged from 1 to 13, with an average of 7 for both groups. The branches that did have several levels generally delineated procedures rather than linked concepts. Integration of concepts, as shown by number of cross links, virtually did not exist. I found only two instances of cross links that showed an important connection, such as the inverse relationship of differentiation and integration. I had shown the students examples of cross links and had stressed their importance as part of the instruction on concept maps. While several students drew cross links, most were trivial, for example, "**variables** can be **letters**."

Complete analysis of the students maps required comparison with the experts' maps. Again, I have chose to use the experts' restricted maps, since they are much closer to the students' maps in content. Inasmuch as the experts are all PhD's in mathematics, I assumed all their concepts and propositions to be valid and relevant.

Unlike many of the students' maps, the experts' maps showed *no hint* of algorithms. Instead, they reflected many categorical groupings, several of which I discussed in detail in the section above on core lists. *None* of the experts demonstrates the students' propensity to think of a function as an equation. Instead, they define it as a correspondence, a map, a pairing, or a rule. *All* incorporate a definition in their map. Five experts (3 traditional, 2 reform) give real-world examples or allude to them. Looking at the overall content and complexity, the experts' maps as a group show much more homogeneity than the students'.

Distinguishing between the two expert groups' concept maps is difficult. As I have noted, all experts referenced the definition and none gave algorithmic groupings, which points to homogeneity rather than divergence. In looking at the structure of the maps, one notes that two experts from each group drew hierarchical maps. One expert from each group drew a spider map, one expert from the traditional group used an unstructured web, and one expert from the reform group used a narrative style. (The expert who used the narrative style told me when I gave him his concept map instruction that he had done concept maps before in a curriculum development setting. His map more closely reflects that setting than the example maps I showed him.) I redrew all but the narrative map in a hierarchical style, realizing I might well be making inferences the experts never intended. The redrawn maps showed three to five levels of hierarchy with no distinctions between the two expert groups. They also showed that two of the reform experts' maps had a large number of cross links, 14 and 17, while the three traditional experts' maps with cross links had a total of 11 between them (5, 5, and 1).

To summarize, a qualitative analy. s of the maps as a whole did indicate two differences between the reform and traditional student groups. The traditional group's maps displayed more and larger algorithmic portions. The traditional students' maps virtually all pointed to an equation view of function. while a good number of the reform students allowed other viewpoints. The reform students may also have a different, broader view of function in real life The differences between the students' maps taken altogether and the experts' maps altogether were much more striking. The reform experts were unanimous in their lack of algorithmic portions while 39% of the student maps included algorithmic groupings. All the expert maps referenced the definition, but only 43% of the students gave terms that could be loosely considered as referring to

the definition. The expert maps contained higher-level categories and were more homogeneous than the students maps, which contained trivial and widely varied groupings. Comparing the reform experts' maps to the traditional experts' maps yielded no major differences.

Implications for concept maps as research tools

A major purpose of this study was to explore the use of the concept map as a research tool in the area of mathematics, particularly as it reflects conceptual understanding. The degree to which concept maps describe a person's actual mental representation is, of course, impossible to know. Nevertheless, the general homogeneity of the experts' maps and their distinct variance from the students' maps lend credibility to the conclusion that concept maps do capture a representative sample of one's conceptual knowledge.

The concept maps in this study shed some light on the issue of hierarchy in knowledge representation. As mentioned in the literature review, Novak and Gowin (1984) require hierarchical maps, since they base their maps on Ausubel's (1968) hierarchical view of knowledge. Other researchers (Hiebert & Lefevre, 1986; White & Gunstone, 1992) allow for hierarchy. For White and Gunstone, certain knowledge domains are hierarchical while others are not. In this study I gave no hierarchical restraint. Consequently, four of the eight experts did **not** create a hierarchical map but rather made a web or spider map. Since the experts did not all draw hierarchical maps, it appears function is not inevitably a hierarchical domain. Yes, one can redraw the web maps in a hierarchical fashion, but such a process necessarily requires the researcher to make her own assumptions about the relative levels of hierarchy. These second-party assumptions may not be consistent with intentions or perceptions of the original map maker.

Some might argue that the diversity in the experts' maps resulted from inadequate instruction on how to draw concept maps. Rogers' (in press) study speaks to this point. Despite detailed instruction on hierarchical maps to students who had drawn 20 concept maps prior to the study, Rogers reports the students' maps showed three patterns of conceptual organization: hierarchical, spider (she terms them "branching"), and random. It appears the students' personal organization of knowledge took precedence over the researcher's prolonged, explicit instruction. I would argue the subjects in my study received adequate instruction for their level of expertise and that differences in the maps' organization reflect their personal views.

Before concept maps become an accepted research tool in mathematics, researchers must resolve issues concerning numerical scoring schemes. Judging from the data I collected and from other studies that have used concept maps, the hierarchical constraint provides one important key to creating a valid scoring scheme for concept maps. Another key is whether one gives the subjects several concepts to use in the maps or merely supplies one main topic. Stuart (1985) describes this process of creating a map from a single given topic as "constructing maps *de novo*." The six studies (Beyerbach, 1986; Coleman, 1993; Laturno, 1994; Markham et al., 1994; Park, 1993; & Wallace & Mintzes (1990)) most relevant to this study all worked in the Novak and Gowin tradition, required hierarchical maps, and gave the maps a numerical score. Beyerbach (1986) is the only one who used concept mapping *de novo*, and her scoring scheme is different from most research in the Novak and Gowin tradition. I did not find her scoring scheme to be valid for measuring the degree of knowledge differentiation. (For more details, see Williams, in press.)

None of the studies convinced me that valid and reliable scoring schemes are currently available for concept maps constructed *de novo*. It does appear

that training students on hierarchical maps, giving them core concepts with which to build their maps, and limiting the subject domain can create a setting in which numerical schemes are valid. Yet in my view, each of these practices reduces the usefulness of concept mapping as a representation of a subject's own personal knowledge structure.

As with all research tools, the concept map has limitations. Concept maps created *de novo* display so much diversity that they are difficult to score numerically, particularly if they do not exhibit a hierarchical structure. White and Gunstone (1992) liken scoring non-hierarchical maps to scoring an essay--one looks for the overall point of view rather than specific knowledge content. If a researcher's aim is primarily to gather quantitative evidence to support an hypothesis, a more structured concept map task can be appropriate. For example, a task where one gives subjects terms to interrelate is better if the researcher is particularly interested in how subjects see the connections between those particular terms. However, if one is looking for an individual's knowledge structure in its "purest" most personal form, the unstructured mapping task is better because it does not put concepts into the subjects' heads. In this study, where concept maps on function were used for the first time, the unstructured task proved appropriate and provided useful information about the broad topic.

The results of this study suggest several areas relating to concept maps where further research is needed. One is the area of knowledge hierarchy. A second is numerical scoring schemes. Would training the experts and students to draw hierarchical maps substantially alter the findings? The researcher could teach the experts to perform the restricted task (relating to first year calculus concepts) by drawing hierarchical maps. One could then compare those hierarchical maps to the ones drawn for this study without the hierarchical

stipulation. A variation on the same study would give the experts the 25 concepts on the combined experts' core list and ask them to draw a concept map limited to these concepts. This could be done with or without the hierarchical constraint. One could study how the maps differed from the *de novo* expert maps drawn for this study, determining if the experts organized their concepts differently or if they added or deleted categories. One could ascertain whether these maps lend themselves to a scoring scheme. And would experts draw hierarchical maps as instructed or would they choose to ignore that instruction as the subjects in Rogers' (in press) study did?

Stuart (1985) states the need for a more holistic and qualitative scoring technique in order to make maps created *de novo* more useful in research. I think concept maps created *de novo* can be useful in research without a numerical scoring technique. While numerical scoring helps compare different groups, *de novo* concept maps as drawn also show major differences between groups. Numerical scoring may conceal the idiosyncratic nature of the maps. Often, researchers find new ideas and relationships in these idiosyncratic maps. As the mathematics community continues to promote meaningful learning and to look for conceptual links, the concept map places an appropriate tool at the researcher's disposal.

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